

# Inverse of a Matrix

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# Inverse of a Matrix

- What is the inverse of a matrix?
- Elementary matrix
- What kinds of matrices are invertible
- Find the inverse of a general invertible matrix

# Inverse of a Matrix

What is the inverse  
of a matrix?

# Inverse of Function

- Two function  $f$  and  $g$  are inverse of each other ( $f=g^{-1}$ ,  $g=f^{-1}$ ) if .....

For *any*  $v$

$$y = g(x) \leftarrow \boxed{g} \leftarrow x = f(v) \leftarrow \boxed{f} \leftarrow v$$

$y = v$

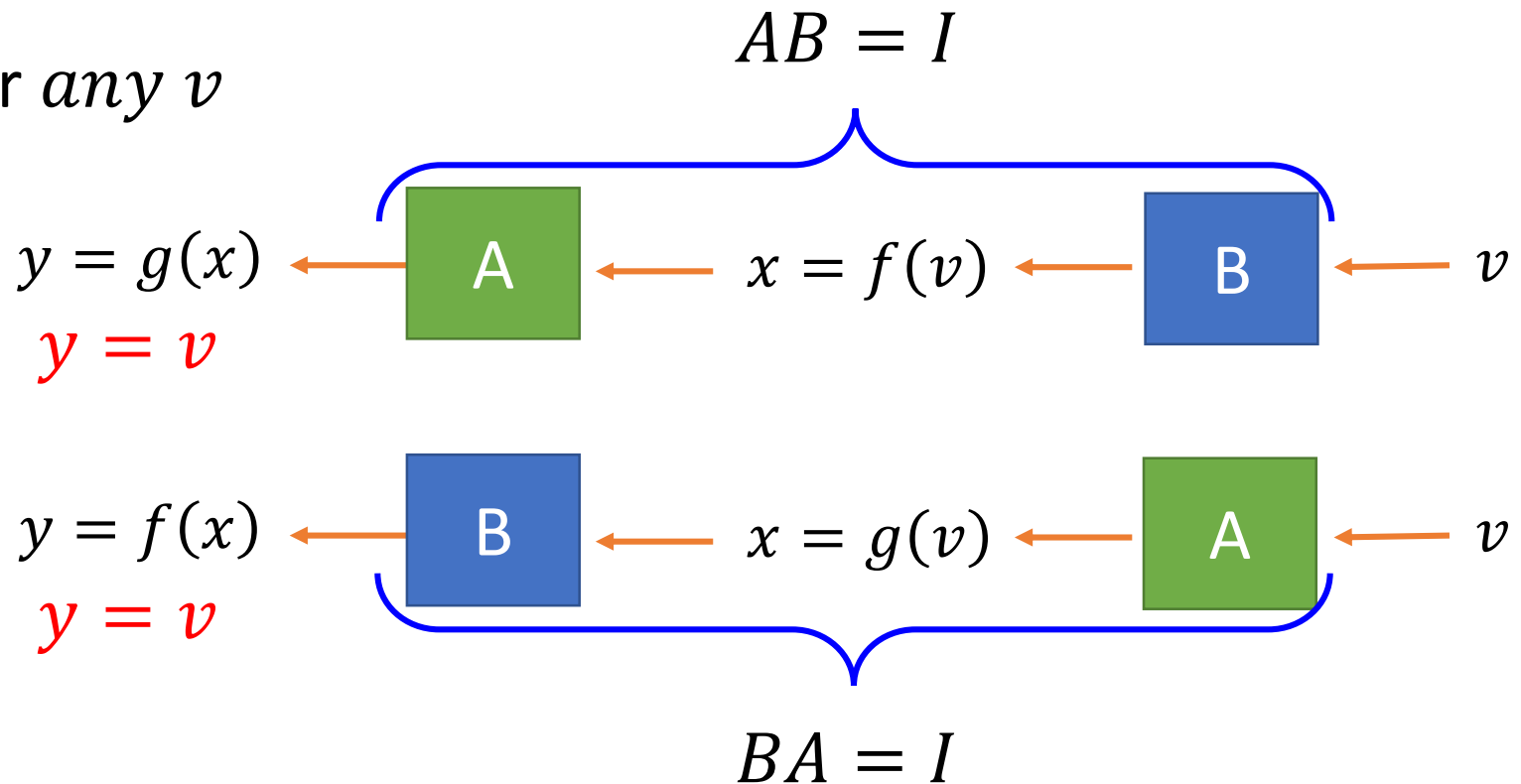
$$y = f(x) \leftarrow \boxed{f} \leftarrow x = g(v) \leftarrow \boxed{g} \leftarrow v$$

$y = v$

# Inverse of Matrix

- If  $B$  is an inverse of  $A$ , then  $A$  is an inverse of  $B$ , i.e.,  $A$  and  $B$  are inverses to each other.

For any  $v$



# Inverse of Matrix

- If  $B$  is an inverse of  $A$ , then  $A$  is an inverse of  $B$ , i.e.,  $A$  and  $B$  are inverses to each other.

$A$  is called invertible if there is a matrix  $B$  such that  $AB = I$  and  $BA = I$

$B$  is an inverse of  $A$

$$B = A^{-1} \quad A^{-1} = B$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Inverse of Matrix

- If  $B$  is an inverse of  $A$ , then  $A$  is an inverse of  $B$ , i.e.,  $A$  and  $B$  are inverses to each other.

$n \times n$ ?

$A$  is called invertible if there is a matrix  $B$  such that  $AB = I$  and  $BA = I$

$B$  is an inverse of  $A$

$$B = A^{-1} \quad A^{-1} = B$$

Non-square matrix cannot be invertible

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

# Inverse of Matrix

- Not all the square matrix is invertible

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Unique

$$AB = I \quad BA = I \quad AC = I \quad CA = I$$

$$B = BI = B(AC) = (BA)C = IC = C$$



# Solving Linear Equations

- The inverse can be used to solve system of linear equations.

$$A\mathbf{x} = \mathbf{b}$$

If A is invertible.

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$\begin{array}{rcl} x_1 + 2x_2 & = & 4 \\ 3x_1 + 5x_2 & = & 7 \end{array}$$

$\underbrace{\hspace{10em}}_{Ax = b}$

$$\begin{aligned} \mathbf{x} &= A^{-1}\mathbf{b} \\ &= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix} \end{aligned}$$

However, this method is computationally inefficient.

# Input-output Model

- 假設世界上只有食物、黃金、木材三種資源

	需要食物	需要黃金	需要木材
生產一單位食物	0.1	0.2	0.3
生產一單位黃金	0.2	0.4	0.1
生產一單位木材	0.1	0.2	0.1

$$\begin{array}{c} Cx \\ \left[ \begin{array}{l} 0.1x_1 + 0.2x_2 + 0.1x_3 \\ 0.2x_1 + 0.4x_2 + 0.2x_3 \\ 0.3x_1 + 0.1x_2 + 0.1x_3 \end{array} \right] \\ \text{須投入} \end{array} = \begin{array}{c} C \\ \left[ \begin{array}{l} 0.1 \quad 0.2 \quad 0.1 \\ 0.2 \quad 0.4 \quad 0.2 \\ 0.3 \quad 0.1 \quad 0.1 \end{array} \right] \\ \text{Consumption} \\ \text{matrix} \end{array} \begin{array}{c} x \\ \left[ \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \right] \\ \text{想生產} \end{array}$$

# Input-output Model

$$\begin{array}{c} Cx \\ \left[ \begin{array}{c} 48 \\ 96 \\ 53 \end{array} \right] \\ \text{須投入} \end{array} = \begin{array}{c} C \\ \left[ \begin{array}{ccc} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{array} \right] \\ \text{Consumption} \\ \text{matrix} \end{array} \begin{array}{c} x \\ \left[ \begin{array}{c} 100 \\ 150 \\ 80 \end{array} \right] \\ \text{想生產} \end{array}$$

須考慮成本：

$$\begin{array}{c} \text{淨收益} \\ x - Cx = \left[ \begin{array}{c} 100 \\ 150 \\ 80 \end{array} \right] - \left[ \begin{array}{c} 48 \\ 96 \\ 53 \end{array} \right] = \left[ \begin{array}{c} 52 \\ 54 \\ 27 \end{array} \right] \end{array} \begin{array}{c} \text{Demand} \\ \text{Vector } d \end{array}$$

# Input-output Model

$$C = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix} \quad d = \begin{bmatrix} 90 \\ 80 \\ 60 \end{bmatrix} \quad \begin{array}{l} \text{Demand} \\ \text{Vector } d \end{array}$$

生產目標  $x$  應該訂為多少?

$$x - Cx = d \quad A = I - C = \begin{bmatrix} 0.9 & -0.2 & -0.1 \\ -0.2 & 0.6 & -0.2 \\ -0.3 & -0.1 & 0.9 \end{bmatrix}$$

$$Ix - Cx = d$$

$$(I - C)x = d$$

$$Ax = b$$

$$b = \begin{bmatrix} 90 \\ 80 \\ 60 \end{bmatrix}$$

$$x = \begin{bmatrix} 170 \\ 240 \\ 150 \end{bmatrix}$$

# Input-output Model

- 提升一單位食物的淨產值，需要多生產多少資源？

Ans: The first column of  $(I - C)^{-1}$

$$(I - C)x = d \quad \longrightarrow \quad x = (I - C)^{-1}d$$

$$d \quad \longrightarrow \quad d + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = d + e_1 \quad \begin{aligned} x' &= (I - C)^{-1}(d + e_1) \\ &= (I - C)^{-1}d + \underline{(I - C)^{-1}e_1} \end{aligned}$$

$$(I - C)^{-1} = \begin{bmatrix} 1.3 & 0.475 & 0.25 \\ 0.6 & 1.950 & 0.50 \\ 0.5 & 0.375 & 1.25 \end{bmatrix}$$

食物    黃金    木材

# Inverse for matrix product

- A and B are invertible nxn matrices, is AB invertible?    yes

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$$

- Let  $A_1, A_2, \dots, A_k$  be nxn invertible matrices. The product  $A_1A_2 \cdots A_k$  is invertible, and

$$(A_1A_2 \cdots A_k)^{-1} = (A_k)^{-1}(A_{k-1})^{-1} \cdots (A_1)^{-1}$$

# Inverse for matrix transpose

- If  $A$  is invertible, is  $A^T$  invertible?

$$(A^T)^{-1} =? (A^{-1})^T$$

$$(AB)^T = B^T A^T$$

$$A^{-1}A = I \implies (A^{-1}A)^T = I \implies A^T(A^{-1})^T = I$$

$$AA^{-1} = I \implies (AA^{-1})^T = I \implies (A^{-1})^T A^T = I$$

Inverse of a Matrix

Inverse of  
elementary matrices



# Elementary Row Operation

- Every elementary row operation can be performed by matrix multiplication.
- 1. Interchange

elementary matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- 2. Scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

- 3. Adding  $k$  times row  $i$  to row  $j$ :

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

# Elementary Matrix


- Every elementary row operation can be performed by matrix multiplication.
- How to find elementary matrix?

elementary matrix

E.g. the elementary matrix that exchange the 1<sup>st</sup> and 2<sup>nd</sup> rows

$$E \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 3 & 6 \end{bmatrix}$$

$$E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiply the 2<sup>nd</sup> row by -4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

# Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad E_1 A =$$

$$E_2 A =$$

$$E_3 A =$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

# Inverse of Elementary Matrix

Reverse elementary row operation

Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Multiply the 2<sup>nd</sup> row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Multiply the 2<sup>nd</sup> row by -1/4

$$E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$



Adding -2 times row 1 to row 3

$$E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

# RREF v.s. Elementary Matrix

- Let  $A$  be an  $m \times n$  matrix with reduced row echelon form  $R$ .

$$E_k \cdots E_2 E_1 A = R$$

- There exists an invertible  $m \times m$  matrix  $P$  such that  $PA=R$

$$P = E_k \cdots E_2 E_1$$

$$P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Inverse of a Matrix

Invertible

# Summary

- Let  $A$  be an  $n \times n$  matrix.  $A$  is invertible if and only if
  - The columns of  $A$  span  $\mathbb{R}^n$
  - For every  $b$  in  $\mathbb{R}^n$ , the system  $Ax=b$  is consistent
  - The rank of  $A$  is  $n$
  - The columns of  $A$  are linear independent
  - The only solution to  $Ax=0$  is the zero vector
  - The nullity of  $A$  is zero
  - The reduced row echelon form of  $A$  is  $I_n$
  - $A$  is a product of elementary matrices
  - There exists an  $n \times n$  matrix  $B$  such that  $BA = I_n$
  - There exists an  $n \times n$  matrix  $C$  such that  $AC = I_n$

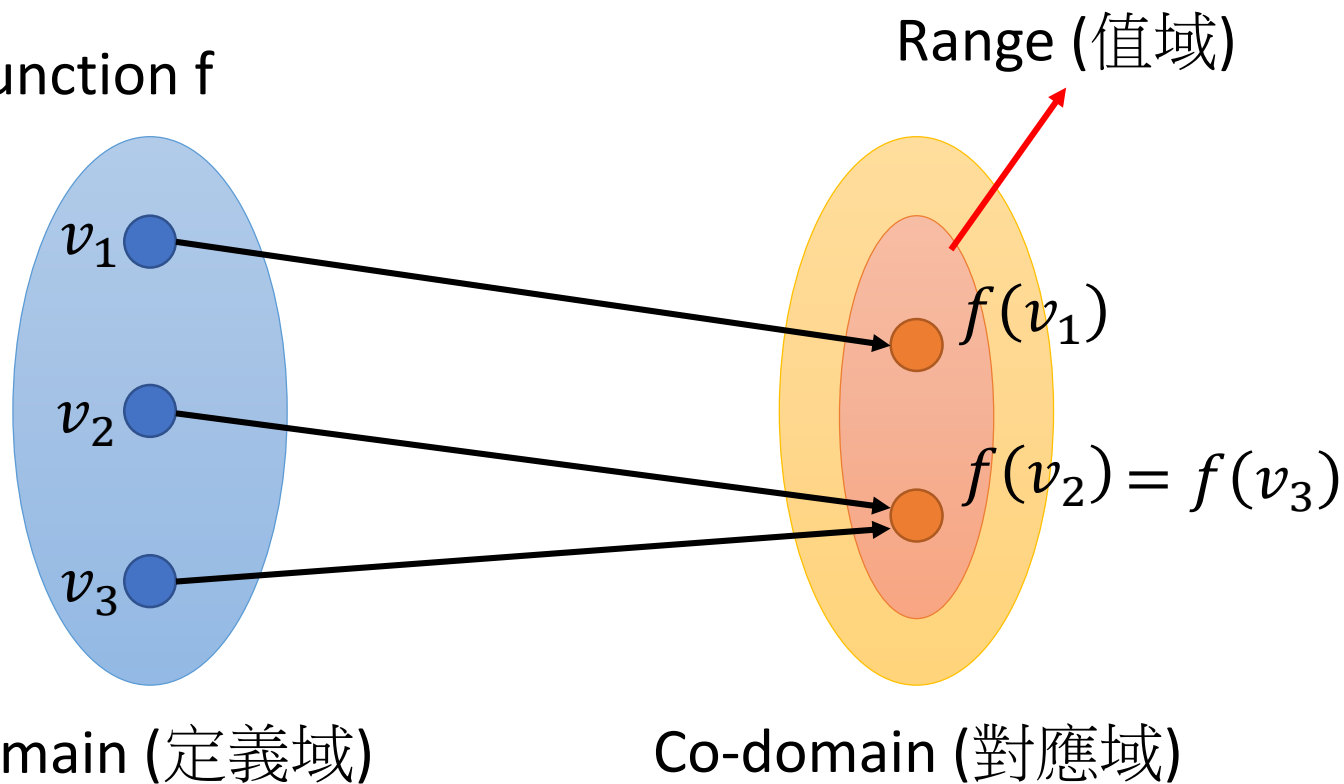




<http://goo.gl/z3J5Rb>

# Review

- Given a function  $f$



Given a linear function corresponding to a  $m \times n$  matrix  $A$

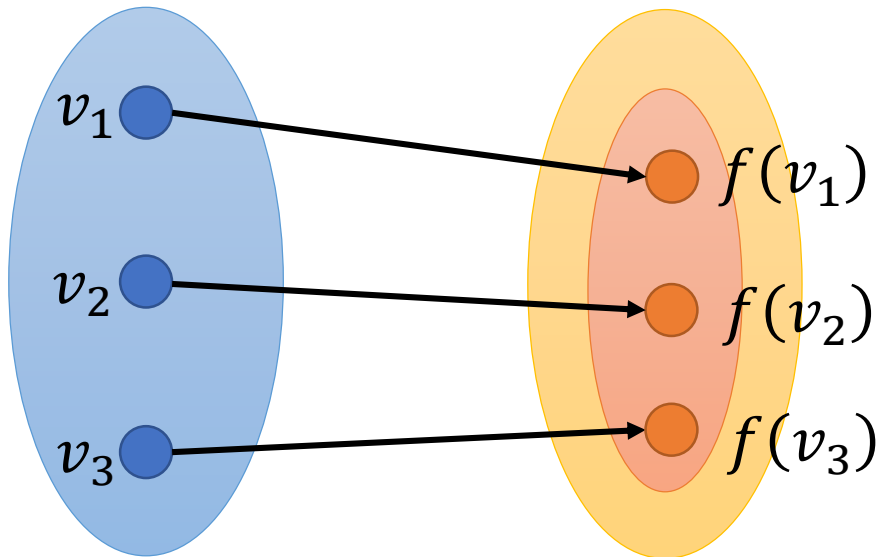
Domain =  $\mathbb{R}^n$

Co-domain =  $\mathbb{R}^m$

Range = ?

# One-to-one

- A function  $f$  is one-to-one



~~$f(x) = b$  has one solution~~

$f(x) = b$  has at most one solution

If co-domain is “smaller” than the domain,  $f$  cannot be one-to-one.

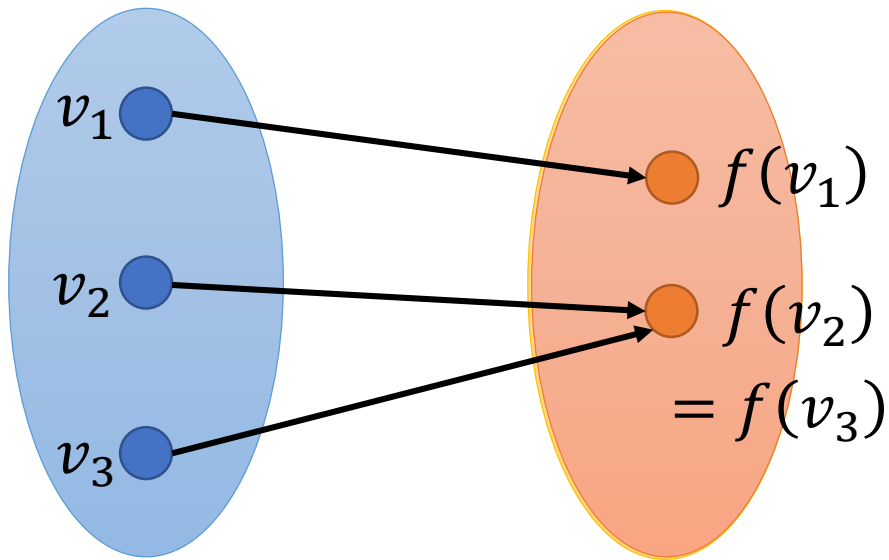
If a matrix  $A$  is 矮胖, it cannot be one-to-one.

The reverse is not true.

If a matrix  $A$  is one-to-one, its columns are independent.

# Onto

- A function  $f$  is onto



Co-domain = range

$f(x) = b$  always have solution

If co-domain is “larger” than the domain,  $f$  cannot be onto.

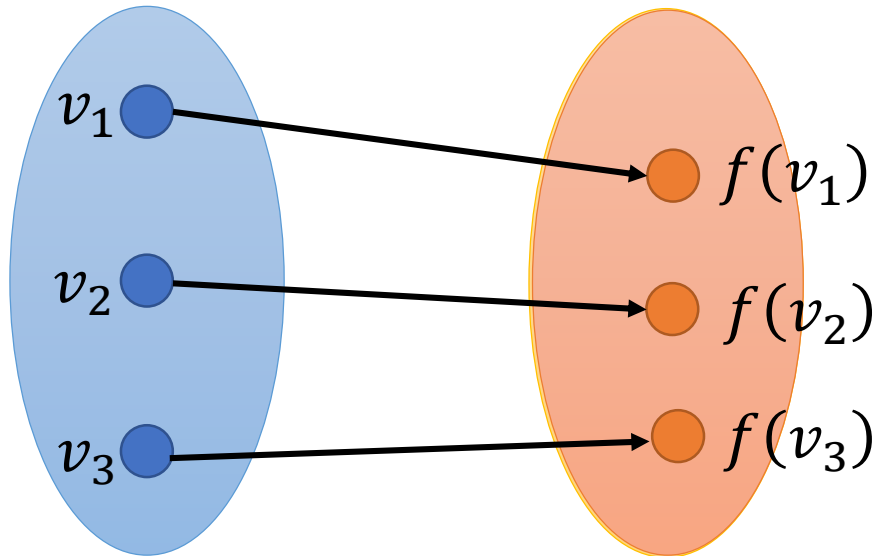
If a matrix  $A$  is 高瘦, it cannot be onto.

The reverse is not true.

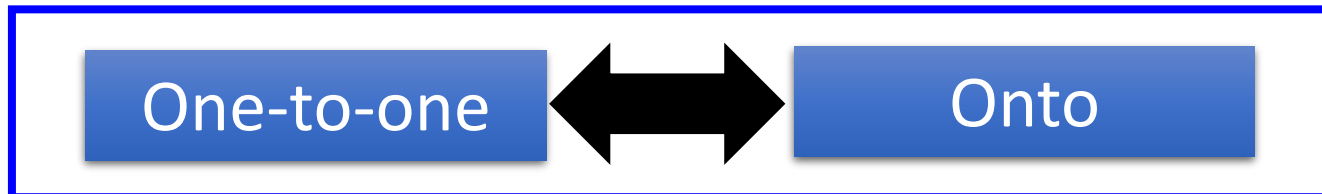
If a matrix  $A$  is onto,  $\text{rank } A = \text{no. of rows}$ .

# One-to-one and onto

- A function  $f$  is one-to-one and onto



The domain and co-domain must have “the same size”.  
The corresponding matrix  $A$  is square.

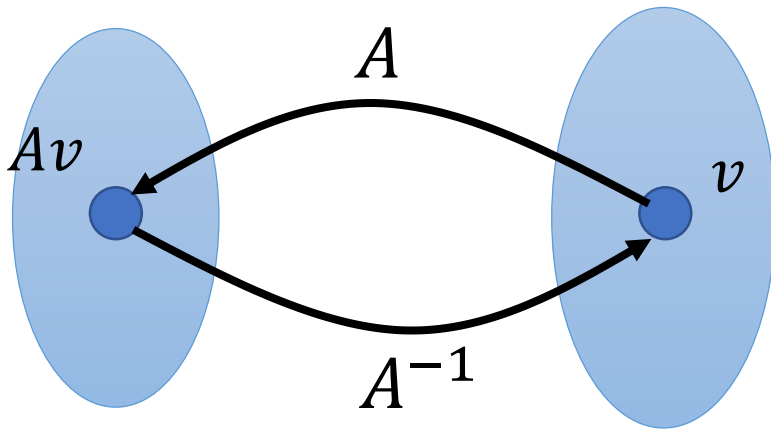


在滿足 Square 的前提下，要就都成立，要就都不成立

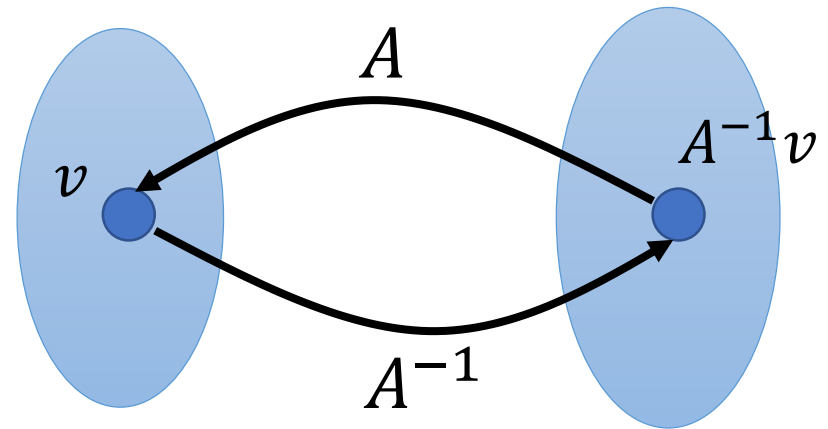
# Invertible

An invertible matrix  $A$  is always square.

- $A$  is called invertible if there is a matrix  $B$  such that  $AB = I$  and  $BA = I$  ( $B = A^{-1}$ )



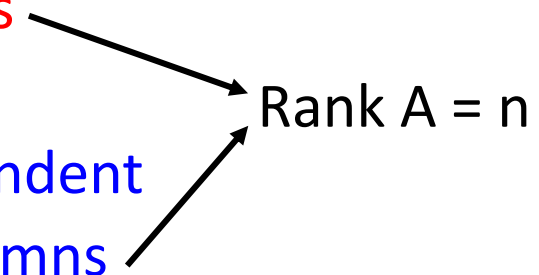
$A$  must be one-to-one



$A$  must be onto

(不然  $A^{-1}$  的 input 就會有限制)

# Invertible

- Let  $A$  be an  $n \times n$  matrix.
    - Onto  $\rightarrow$  One-to-one  $\rightarrow$  invertible
      - The columns of  $A$  span  $\mathbb{R}^n$
      - For every  $b$  in  $\mathbb{R}^n$ , the system  $Ax=b$  is consistent
      - The rank of  $A$  is the number of rows
    - One-to-one  $\rightarrow$  Onto  $\rightarrow$  invertible
      - The columns of  $A$  are linear independent
      - The rank of  $A$  is the number of columns
      - The nullity of  $A$  is zero
      - The only solution to  $Ax=0$  is the zero vector
      - The reduced row echelon form of  $A$  is  $I_n$
- Rank  $A = n$
- 

# Invertible

- Let  $A$  be an  $n \times n$  matrix.  $A$  is invertible if and only if
  - The reduced row echelon form of  $A$  is  $I_n$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} I_n \quad \text{Invertible}$$

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Not Invertible}$$



# Summary

- Let  $A$  be an  $n \times n$  matrix.  $A$  is invertible if and only if

onto

- The columns of  $A$  span  $\mathbb{R}^n$
- For every  $b$  in  $\mathbb{R}^n$ , the system  $Ax=b$  is consistent

- The rank of  $A$  is  $n$

||

square  
matrix

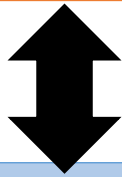
One-to-one

- The columns of  $A$  are linear independent
- The only solution to  $Ax=0$  is the zero vector
- The nullity of  $A$  is zero
- The reduced row echelon form of  $A$  is  $I_n$

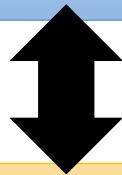
- $A$  is a product of elementary matrices
- There exists an  $n \times n$  matrix  $B$  such that  $BA = I_n$
- There exists an  $n \times n$  matrix  $C$  such that  $AC = I_n$

# Invertible

An  $n \times n$  matrix  $A$  is invertible.



The reduced row echelon form of  $A$  is  $I_n$



$A$  is a product of elementary matrices

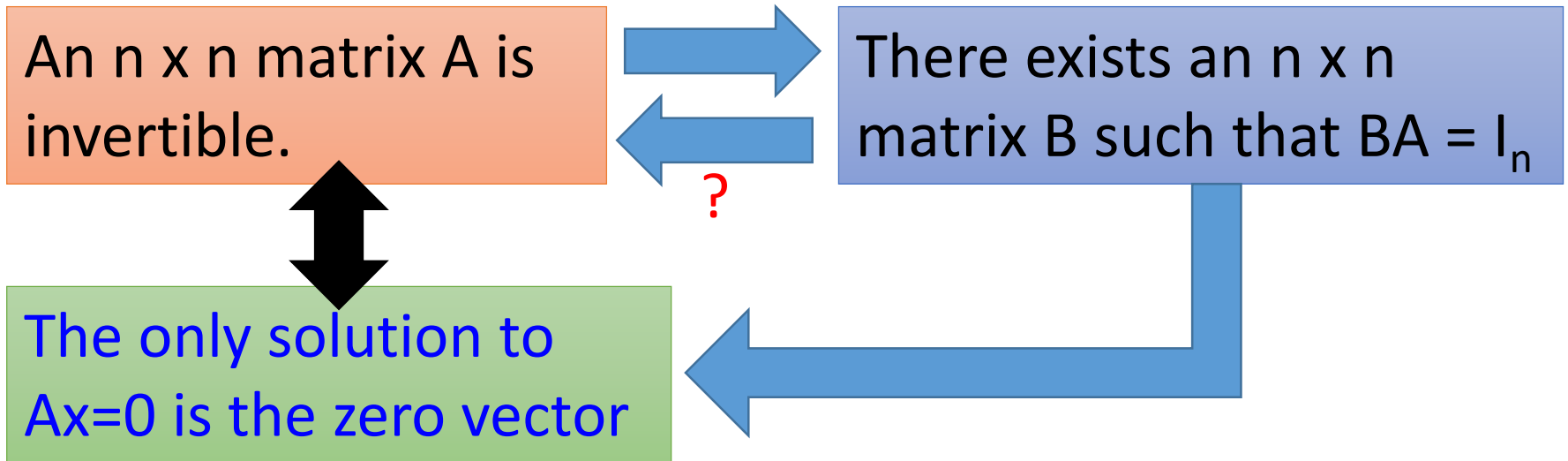
$$R = \text{RREF}(A) = I_n$$

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

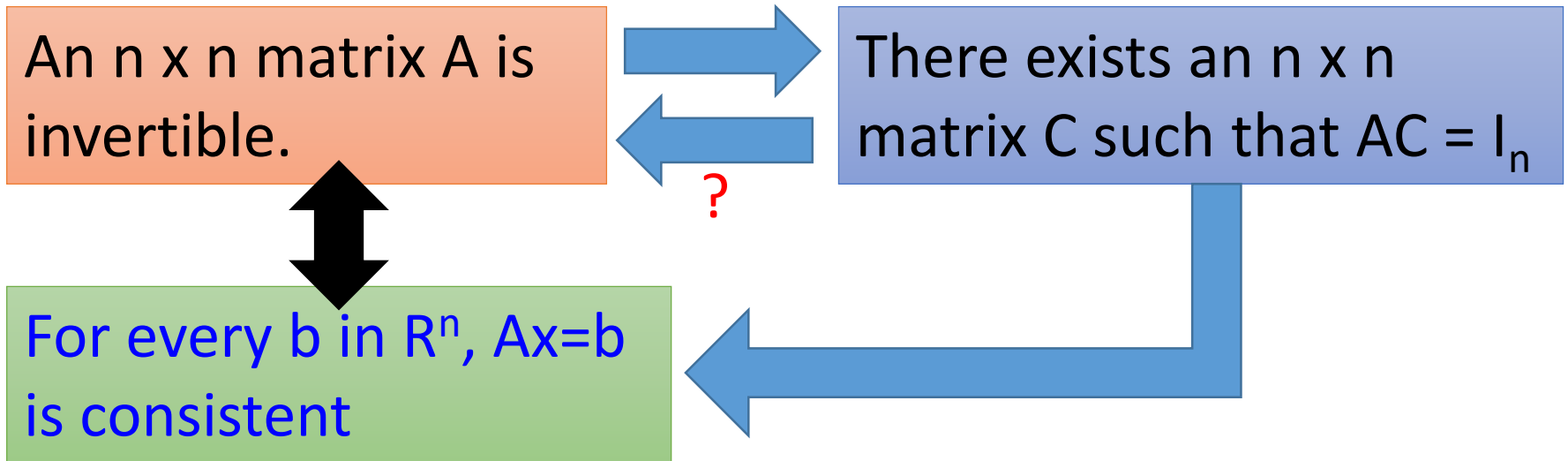
# Invertible



If  $Av = 0$ , then ....

$$\begin{array}{ccc} \underline{BA} = \underline{I_n} & \longrightarrow & v = 0 \\ \swarrow \quad \searrow & & \\ BA v = 0 & I_n v = v & \end{array}$$

# Invertible



For any vector  $b$ ,

$$\begin{array}{ccc} & AC = I_n & \\ & \swarrow \quad \searrow & \\ ACb & & I_n b = b \end{array} \quad \longrightarrow \quad Cb \text{ is always a solution for } b$$

# Summary

- Let  $A$  be an  $n \times n$  matrix.  $A$  is invertible if and only if

onto

- The columns of  $A$  span  $\mathbb{R}^n$
- For every  $b$  in  $\mathbb{R}^n$ , the system  $Ax=b$  is consistent

- The rank of  $A$  is  $n$

||

square  
matrix

One-to-one

- The columns of  $A$  are linear independent
- The only solution to  $Ax=0$  is the zero vector
- The nullity of  $A$  is zero
- The reduced row echelon form of  $A$  is  $I_n$

- $A$  is a product of elementary matrices
- There exists an  $n \times n$  matrix  $B$  such that  $BA = I_n$
- There exists an  $n \times n$  matrix  $C$  such that  $AC = I_n$

# Questions

- If  $A$  and  $B$  are matrices such that  $AB=I_n$  for some  $n$ , then both  $A$  and  $B$  are invertible.
- For any two  $n$  by  $n$  matrices  $A$  and  $B$ , if  $AB=I_n$ , then both  $A$  and  $B$  are invertible.

# Inverse of a Matrix

Inverse of

General invertible matrices

# 2 X 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad \text{Find } e, f, g, h$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $ad - bc = 0$ ,  $A$  is not invertible.



# Algorithm for Matrix Inversion

- Let  $A$  be an  $n \times n$  matrix.  $A$  is invertible if and only if
  - The reduced row echelon form of  $A$  is  $I_n$

$$\underline{E_k \cdots E_2 E_1} A = R = I_n$$
$$A^{-1}$$

$$A^{-1} = E_k \cdots E_2 E_1$$

# Algorithm for Matrix Inversion

- Let  $A$  be an  $n \times n$  matrix. Transform  $[A \ I_n]$  into its RREF  $[R \ B]$ 
  - $R$  is the RREF of  $A$
  - $B$  is an  $n \times n$  matrix (not RREF)
- If  $R = I_n$ , then  $A$  is invertible
  - $B = A^{-1}$

$$\begin{aligned} & E_k \cdots E_2 E_1 [A \ I_n] \\ &= \begin{bmatrix} \underline{R} & \underline{E_k \cdots E_2 E_1} \\ I_n & A^{-1} \end{bmatrix} \end{aligned}$$

# Algorithm for Matrix Inversion

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} I_n \quad \text{Invertible}$$

$$\left[ A \quad I_3 \right] = \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right]$$

$A^{-1}$

# Algorithm for Matrix Inversion

- Let  $A$  be an  $n \times n$  matrix. Transform  $[A \ I_n]$  into its RREF  $[R \ B]$ 
  - $R$  is the RREF of  $A$
  - $B$  is a  $n \times n$  matrix (not RREF)
- If  $R = I_n$ , then  $A$  is invertible
  - $B = A^{-1}$
- To find  $A^{-1}C$ , transform  $[A \ C]$  into its RREF  $[R \ C']$ 
  - $C' = A^{-1}C$

$$E_k \cdots E_2 E_1 [A \ C] = \begin{bmatrix} R & \overbrace{E_k \cdots E_2 E_1 C}^{A^{-1}C} \\ I_n & \end{bmatrix}$$

# Appendix

# 2 X 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a \\ c \end{bmatrix} \neq k \begin{bmatrix} b \\ d \end{bmatrix} \quad \frac{a}{b} \neq \frac{c}{d}$$

$$ad \neq bc \quad ad - bc \neq 0$$

$$A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad \text{Find } e, f, g, h$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$